THERMAL RESISTANCE BETWEEN TWO LONG PARALLEL CYLINDERS OF ELLIPTICAL CROSS SECTION

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It is shown that the problem of determining the thermal resistance between two long parallel cylinders of elliptical cross section can be reduced to the corresponding problem for two infinitely thin parallel bands. The thermal resistance of the system is plotted vs the geometrical dimensions.

1. The problem of calculating the thermal resistance between two long parallel cylinders of elliptical cross section situated in a homogeneous medium (Fig. 1) has a known solution for the limiting cases shown in Fig. 2.

In the first two cases (Fig. 2a, b), the thermal resistance R per unit length in longitudinal direction is

$$R = \frac{1}{\lambda} \frac{K}{K'}, \tag{1}$$

where λ is the heat conductivity coefficient of the ambient medium, and K and K' are complete elliptic integrals of the first kind, whose moduli k and k' are defined by the equations:

$$K'E'(\gamma, k) - E'F'(\gamma, k) = \pi \frac{a'}{b'}, \tag{2}$$

$$\sin^2 \gamma = \frac{K' - E'}{(1 - k^2) K'}, \tag{3}$$

$$k' = \sqrt{1 - k^2} \tag{4}$$

for the configuration shown in Fig. 2a, and by the formulas

$$k = \frac{d' - 2a'}{d' + 2a'}, \quad k' = \sqrt{1 - k^2} \tag{5}$$

for the configuration shown in Fig. 2b. In formulas (2) and (3), K', E', F'(γ, k) are the notations for the incomplete elliptic integrals of the second kind with modulus k'.

For the third case (Fig. 2c), we have

$$R = \frac{1}{\pi\lambda} \operatorname{Ar ch} \frac{d}{2a}. \tag{6}$$

The general case of elliptical cylinders with an arbitrary ratio of the semiaxes a and b (Fig. 1), apparently, has not been examined. In the following, it will be shown that by appropriate conformal mapping, this case can be reduced to one of the degenerate cases shown in Fig. 2.

2. We known from [1] that a multiply connected region D in the plane $z = x + iy$, bounded from the inside m by smooth contours l_k, can be transformed into a plane $\zeta = \xi + i\eta$ with m slots parallel to the real axis. The mapping function, in this case, has the form

$$\zeta = f(z) = z - \sum_{k=1}^{m} \frac{1}{2\pi i} \int\limits_{l_k} \frac{\xi_k(t)\, dt}{t - z}, \tag{7}$$

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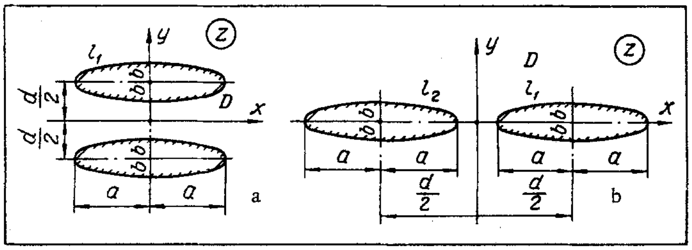

Fig. 1. Symmetrically located elliptical cylinders.

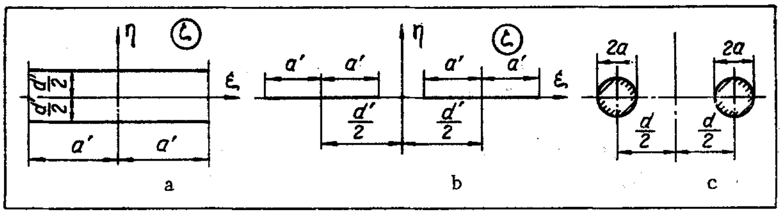

Fig. 2. Elliptic cylinders in the degenerate cases: a) infinitely thin bands; c) circular cylinders.

where t is the value of the variable z at the contour l_k; $\xi_k(t)$ is the real part of the function f(z) at this contour. By making point z in the region D approach point t_0 of the contour l_i, performing the passage to the limit conventionally used for Cauchy integrals [2], and separating the real from the imaginary part, we arrive at the equation

$$\xi_i(t_0) = 2x(t_0) - \sum_{k=1}^{m} \frac{1}{\pi} \int_{l_k} \frac{\xi_k(t) \cos\theta \, dl_k}{r}, \tag{8}$$

where r is the radius vector drawn from point t_0 on contour l_i to a point on the contour l_k, and θ is the angle formed by this radius vector and the normal to contour l_k directed toward the interior of region D; the integral over contour l_i is treated as singular in Cauchy's sense.

In our case of a doubly connected region D whose boundaries are identical symmetrically situated ellipses (Fig. 1), the corresponding slots in the plane ζ are located also symmetrically (Fig. 2a and b). Hence, of the two equations of type (8), it is sufficient to examine one equation:

$$\xi_1(t_0) = 2x(t_0) - \frac{1}{\pi} \int_{l_1} \frac{\xi_1(t) \cos\theta \, dl_1}{r} - \frac{1}{\pi} \int_{l_2} \frac{\xi_2(t) \cos\theta \, dl_2}{r}, \tag{9}$$

where, for the configuration shown in Fig. 1a, $\xi_2(x) = \xi_1(x)$, while for the configuration shown in Fig. 1b, $\xi_2(-x) = -\xi_1(x)$. Solving Eq. (9), and substituting ξ_1 and ξ_2 into (7), we obtain the mapping function f(z). From the real part of this function, one can obtain the values of ξ at the end points of the slots, and from the imaginary part, the values of η_1 and η_2 at these slots. In this way, both the length of the slots and their position in the plane ζ will be determined. Since the procedure for solving the problem is almost the same in each of the cases shown in Fig. 1, we may limit the analysis to one of them – the case shown in Fig. 1a.

3. The integral equation (9) is similar to the equation of the corresponding two-dimensional problem in hydrodynamics [3], and lends itself to solution in the same way. By representing the ellipse equations in parametric form:

$$x = a\cos\varphi, \quad y = \frac{d}{2} + b\sin\varphi;$$

$$x = a\cos\psi, \quad y = -\frac{d}{2} + b\sin\psi,$$

the expressions in the integrand of Eq. (9) can be reduced to the form

$$\frac{\xi_1 \cos\theta\, dl_1}{r} = \xi_1(\varphi)\frac{b}{2a}\,\frac{d\varphi}{1-\varepsilon^2\cos^2\dfrac{\varphi+\varphi_0}{2}}, \tag{10}$$

$$\frac{\xi_2 \cos\theta\, dl_2}{r} = \frac{\xi_2(\psi)}{2\delta}\left\{\frac{\beta}{2\delta}\,[1-\cos(\varphi_0-\psi)] - \sin\psi\right\}$$

$$\times\left\{1+\frac{\beta}{\delta}(\sin\varphi_0-\sin\psi)+\frac{1}{4\delta^2}\left[(\cos\varphi_0-\cos\psi)^2+\beta^2(\sin\varphi_0-\sin\psi)^2\right]\right\}^{-1} d\psi, \tag{11}$$

where $\beta = b/a$; $\delta = d/2a$; $\varepsilon = \sqrt{1-\beta^2}$; φ_0 is the value of φ at point t_0. Expression (10) can be represented in the form of a trigonometric series

$$\frac{\xi_1\cos\theta\, dl_1}{r} = \xi_1(\varphi)\left[\frac{1}{2}+\sum_{k=1}^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^k(\cos k\varphi_0\cos k\varphi-\sin k\varphi_0\sin k\varphi)\right] d\varphi, \tag{12}$$

and expression (11) in the form of a series in powers of $1/\delta$. By eliminating the case where δ is close to unity (the case of very closely situated ellipses), and limiting the analysis to terms with $1/\delta^3$, instead of (11), we obtain

$$\frac{\xi_2\cos\theta\, dl_2}{r} = \frac{\xi_2(\psi)}{2\delta}\left\{a_0+\frac{a_1-a_0b_1}{\delta}+\frac{1}{\delta^2}\,[a_0(b_1^2-b_2)-a_1b_1]\right\} d\psi, \tag{13}$$

where

$$a_0 = -\sin\psi;\quad b_1=\beta(\sin\varphi_0-\sin\psi);$$
$$a_1=\frac{\beta}{2}\,[1-\cos(\varphi_0-\psi)];$$
$$b_2=\frac{1}{4}\,[(\cos\varphi_0-\cos\psi)^2+\beta^2(\sin\varphi_0-\sin\psi)^2].$$

It can be readily seen that the quantities $\xi_1(\varphi)$, $\xi_1(\varphi_0)$, and $\xi_2(\psi)$ are identical functions of their arguments, and therefore can be represented by identical Fourier series. By virtue of the parity of functions $\xi_1(\varphi)$ and $\xi_2(\varphi)$ and the self-evident relations

$$\xi_1(\varphi) = -\xi_1(\pi-\varphi);\quad \xi_1(\varphi_0) = -\xi_1(\pi-\varphi_0);$$
$$\xi_2(\psi) = -\xi_2(\pi-\psi)$$

these series have the form:

$$\xi_1(\varphi)=\sum_{n=0}^{\infty}A_{2n+1}\cos(2n+1)\varphi,$$
$$\xi_1(\varphi_0)=\sum_{n=0}^{\infty}A_{2n+1}\cos(2n+1)\varphi_0, \tag{14}$$
$$\xi_2(\psi)=\sum_{n=0}^{\infty}A_{2n+1}\cos(2n+1)\psi.$$

By substituting (12) (13), and (14) into the basic equation (9), performing integration, and equating the coefficients in front of the cosines of like arcs, we obtain

$$A_1 = 2a - \frac{1-\beta}{1+\beta}A_1+\frac{\beta}{4\delta^2}A_1,\quad A_3=0 \text{ and so forth,}$$

whence

$$A_1=\frac{a+b}{1-h}=a\,\frac{1+\beta}{1-h}, \tag{15}$$

where

$$h=\frac{\beta(1+\beta)}{8\delta^2}, \tag{16}$$

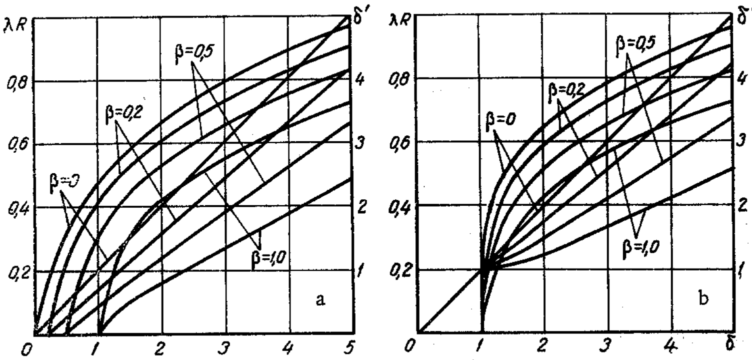

Fig. 3. Curves for determining the thermal resistance between the cylinders shown in Fig. 1a (a) and 1b (b).

and hence we may write

$$\xi_1(\varphi) = A_1 \cos \varphi; \quad \xi_2(\psi) = A_1 \cos \psi. \tag{17}$$

These expressions are first approximations for functions $\xi_1(\varphi)$ and $\xi_2(\varphi)$. In order to obtain the second approximation, one must substitute (17) into the basic Eq. (7) and integrate. As shown in the appendix, as a result of these operations we obtain the function

$$\zeta = f(z) = z + \frac{A_1 b}{a^2 - b^2}\left[2z - \sqrt{\left(z - i\frac{d}{2}\right)^2 - a^2 + b^2} - \sqrt{\left(z + i\frac{d}{2}\right)^2 - a^2 + b^2}\right], \tag{18}$$

which for points on ellipse 1 (for $z = t_1$) may be approximated in the form

$$f(t_1) = \xi_1 + i\eta_1 = \frac{a + b + bg}{a} x_1 + \frac{(1+g)(a+b)bx_1}{2\left[x_1^2 + \left(\frac{d}{2} + y_1\right)^2\right]} + i\left\{\frac{d}{2} - g\left(y_1 - \frac{d}{2}\right) - \frac{(1+g)(a+b)b\left(\frac{d}{2} + y_1\right)}{2\left[x_1^2 + \left(\frac{d}{2} + y_1\right)^2\right]}\right\}, \tag{19}$$

where $g = h/(1 - h)$.

Since formula (19) yields only an approximate expression for function $f(t_1)$, the imaginary part of this function does not remain constant when point t_1 displaces itself along ellipse 1, while the corresponding point in the plane ψ is displaced, not along a straight line parallel to the abscissa, but along the curve (19) shaped as a narrow loop elongated in direction of the ξ axis. The smaller b as compared to a and a as compared to d, the closer this loop approaches the required slot. Limiting ourselves to the approximation obtained, we can determine the length of the slot $2a'$ as the difference between the values of ξ_1 for $x_1 = a$ and $x_1 = -a$, while the spacing d' between the slots can be determined as the sum of the values of η_1 for $y_1 = (d/2) + b$ and $y_1 = (d/2) - b$. In this way, we obtain

$$2a' = 2A_1\left[1 - \frac{\beta}{8\delta^2(4\delta^2 + 1)}\right], \tag{20}$$

$$d' = d\left[1 - \frac{2g}{1 - \frac{\beta^2}{4\delta^2}}\right]. \tag{21}$$

It can be shown that when the condition

$$\delta^2 > (1 + \beta)\beta, \quad \text{i. e.,} \quad d^2 > 4(a + b)b, \tag{22}$$

which excludes the case of closely situated ellipses, is fulfilled, the second approximation does not introduce any appreciable improvements to the values of a' and d'.*

*Thus, for $\beta = 1$ and $\delta^2 = 2$, the next approximation yields values for a'' and d'' which differ by less than 1.5% from the values for a' and d'.

Figure 3a shows plots of $\delta' = d'/2a'$ vs $\delta = d/2a$ for various values of $\beta = b/a$. For values of δ and β that satisfy condition (22), the curves are plotted from formulas (20) and (21). The initial portions of the curves are obtained by interpolation, allowing that $\delta' = 0$ for $d = 2b$, i.e., for $\delta = \beta$.

The formulas (20), (21) and the curves in Fig. 3a define the geometrical dimensions of the systems of bands that are equivalent to the elliptical cylinders under consideration, i.e., for which the thermal resistance R is the same as that of the cylinders. By using further the formulas (1)-(4), it is possible to obtain values of R that correspond to given values of a, b, and d. The corresponding curves are plotted in Fig. 3a, as a function of $\delta = d/2a$.

4. In the case of cylinders arranged according to Fig. 3b, the problem is solved in a similar fashion. Limiting the analysis, as before, to the second approximation, for the dimensions of the equivalent system of bands (Fig. 2b), we obtain

$$2a' = 2a\,\frac{1+\beta}{1+h}\left[1 - \frac{\beta}{8\delta^2(4\delta^2-1)}\right], \tag{23}$$

$$d' = d\left[1 + \frac{8g\,\delta^2}{4\delta^2-1}\right], \tag{24}$$

where

$$g = \frac{h}{1+h}, \quad h + \frac{\beta(1+\beta)}{8\delta^2}, \quad \beta = \frac{b}{a}, \quad \delta = \frac{d}{2a}.$$

The curves that correspond to this case are plotted in Fig. 3b as a function of $\delta = d/2a$.

APPENDIX

In order to obtain the second approximation for the mapping function $\zeta = f(z)$, we substitute (17) into the initial expression (7). As a result, we have

$$\zeta = f(z) = z - \frac{A_1}{2\pi i}\int_{l_1}\frac{\cos\varphi}{t-z}\,dt - \frac{A_1}{2\pi i}\int_{l_2}\frac{\cos\psi}{t-z}\,dt. \tag{25}$$

We evaluate the first of the two integrals in this expression

$$J_1 = \frac{1}{2\pi i}\int_{l_1}\frac{\cos\varphi}{t-z}\,dt.$$

By introducing a new variable $w = (z - id)/2$ and correspondingly $\tau = (t - id)/2$, for $\cos\varphi$ we get

$$\cos\varphi = \frac{a}{c^2}\left[\tau - \frac{b}{a}\sqrt{\tau^2 - c^2}\right],$$

where $c^2 = a^2 - b^2$; J_1 may be then represented in the form

$$J_1 = \frac{a}{c^2}\,I_1,$$

where

$$I_1 = \frac{1}{2\pi i}\int_{l_1}\frac{\tau - \frac{b}{a}\sqrt{\tau^2-c^2}}{\tau - w}\,d\tau.$$

Since the integrand has branch points ($\tau = \pm c$) inside the contour l_1, we use the residue theorem in a region external with respect to l_1 for calculating I_1:

$$I_1 = -\operatorname{Res}(\infty) - \operatorname{Res}(w).$$

It can be readily shown that

$$\operatorname{Res}(\infty) = \frac{b-a}{a}\,w; \quad \operatorname{Res}(w) = w - \frac{b}{a}\sqrt{w^2-c^2}.$$

In this way, returning to the variable z, for the integral J_1 we get

$$J_1 = -\frac{b}{c^2}\left[z - i\frac{d}{2} - \sqrt{\left(z - i\frac{d}{2}\right)^2 - c^2}\right].$$

For the second integral, we get in the same way

$$J_2 = -\frac{b}{c^2}\left[z + i\frac{d}{2} - \sqrt{\left(z + i\frac{d}{2}\right)^2 - c^2}\right].$$

By substituting J_1 and J_2 into (25), we arrive at formula (18).

Setting $z = t_1 = x_1 + iy_1$ in (18), representing the first radical in this formula in the form

$$\sqrt{\left(t_1 - i\frac{d}{2}\right)^2 - c^2} = \frac{a}{b}t_1 - \frac{c^2}{ab}x_1 - i\frac{ad}{2b},$$

and the second radical in the form

$$\sqrt{\left(t_1 + i\frac{d}{2}\right)^2 - c^2} = t_1 + i\frac{d}{2} - \frac{c^2}{2\left(t_1 + i\frac{d}{2}\right)} \cdots$$

and taking the first two terms of the series, we obtain formula (19).

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